Leakage theory for unconfined aquifers

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Abstract

A semi-analytical solution is presented for the problem of flow in a system comprising an unconfined aquifer and a confined aquifer that are separated by an aquitard. The unconfined aquifer is pumped continuously at a constant rate from a well of infinitessimal radius that partially penetrates its saturated thickness. The solution is termed semi-analytical since the exact solution is obtained in the double Laplace-Hankel transform space and is then inverted numerically. The solution presented here is more general than similar solutions obtained for confined aquifer flow as we do not adopt the assumption of unidirectional flow in the confined aquifer (typically assumed to be horizontal) and the aquitard (typically assumed to be vertical). Aquitard storage is not neglected.

Mathematical formulation

Assumptions

i. Flow in the unsaturated zone above the unconfined aquifer can be neglected

ii. Flow is radially symmetric

iii. The confined aquifer is bounded from below by a no-flow boundary

iv. The coordinate axes are parallel to the principal directions of hydraulic conductivity

Dimensionless Governing Flow Equation

$$\frac{\partial s_{D,i}}{\partial t_D} = \frac{\alpha_{D,r}^{(i)}}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial s_{D,i}}{\partial r_D}\right) + \alpha_{D,z}^{(i)}\frac{\partial^2 s_{D,i}}{\partial z_D^2} \tag{1}$$

Initial and far-field conditions

$$s_{D,i}(r_D, z_D, 0) = \lim_{r_D\to\infty} s_{D,i}(r_D, z_D, t_D) = 0. \tag{2}$$

Pumping well condition (at $r_D = 0$)

$$\lim_{r_D\to 0} r_D\frac{\partial s_{D,1}}{\partial r_D} = \begin{cases} 0 & \forall z \in (-d_D, 0] \\ -2/(l_D - d_D) & \forall z \in [-l_D, -d_D] \\ 0 & \forall z \in [-1, -l_D) \end{cases} \tag{3}$$

Linearized kinematic condition at water-table (i.e. at $z_D = 0$)

$$-\left.\frac{\partial s_{D,1}}{\partial z_D}\right|_{z_D=0} = \frac{1}{\alpha_{D,Y}}\left.\frac{\partial s_{D,1}}{\partial t_D}\right|_{z_D=0}, \tag{4}$$

No-pumping at aquitard center

$$\lim_{r_D\to 0} r_D\frac{\partial s_{D,2}}{\partial r_D} = 0, \tag{5}$$

No-flow across confined aquifer center and base ($z_D = -b_{D,3}$)

$$\lim_{r_D\to 0} r_D\frac{\partial s_{D,3}}{\partial r_D} = \left.\frac{\partial s_{D,3}}{\partial z_D}\right|_{z_D=-b_{D,3}} = 0. \tag{6}$$

The continuity conditions become at (a) unconfined aquifer-aquitard contact:

$$s_{D,1}(r_D, -1, t_D) = s_{D,2}(r_D, -1, t_D) \tag{7}$$

$$\left.\frac{\partial s_{D,1}}{\partial z_D}\right|_{z_D=-1} = \kappa_{z,2}\left.\frac{\partial s_{D,2}}{\partial z_D}\right|_{z_D=-1} \tag{8}$$

(b) confined aquifer-aquitard contact:

$$s_{D,2}(r_D, -b_{D,2}, t_D) = s_{D3}(r_D, -b_{D,2}, t_D) \tag{9}$$

$$\left.\frac{\partial s_{D,2}}{\partial z_D}\right|_{z_D=-b_{D,2}} = \kappa_{z,3}\left.\frac{\partial s_{D,3}}{\partial z_D}\right|_{z_D=-b_{D,2}}. \tag{10}$$

List of Dimensionless Variables and Parameters

$s_{D,i} = s_i/H_c$, $z_D = z/b_1$, $r_D = r/b_1$, $t_D = \alpha_r t/b_1^2$, $\alpha_r = K_{r,1}/S_s$, $l_D = l/b_1$, $d_D = d/b_1$, $b_{D,2} = b_2/b_1$,$b_{D,3} = b_3/b_1$, $\alpha_{D,Y} = \alpha_Y/\alpha_r$, $\alpha_Y = b_1 K_{z,1}/S_Y$, $\kappa_2 = K_{z,2}/K_{z,1}$, $\kappa_3 = K_{z,3}/K_{z,2}$, $H_c = Q/(4\pi b_1 K_{r,1})$

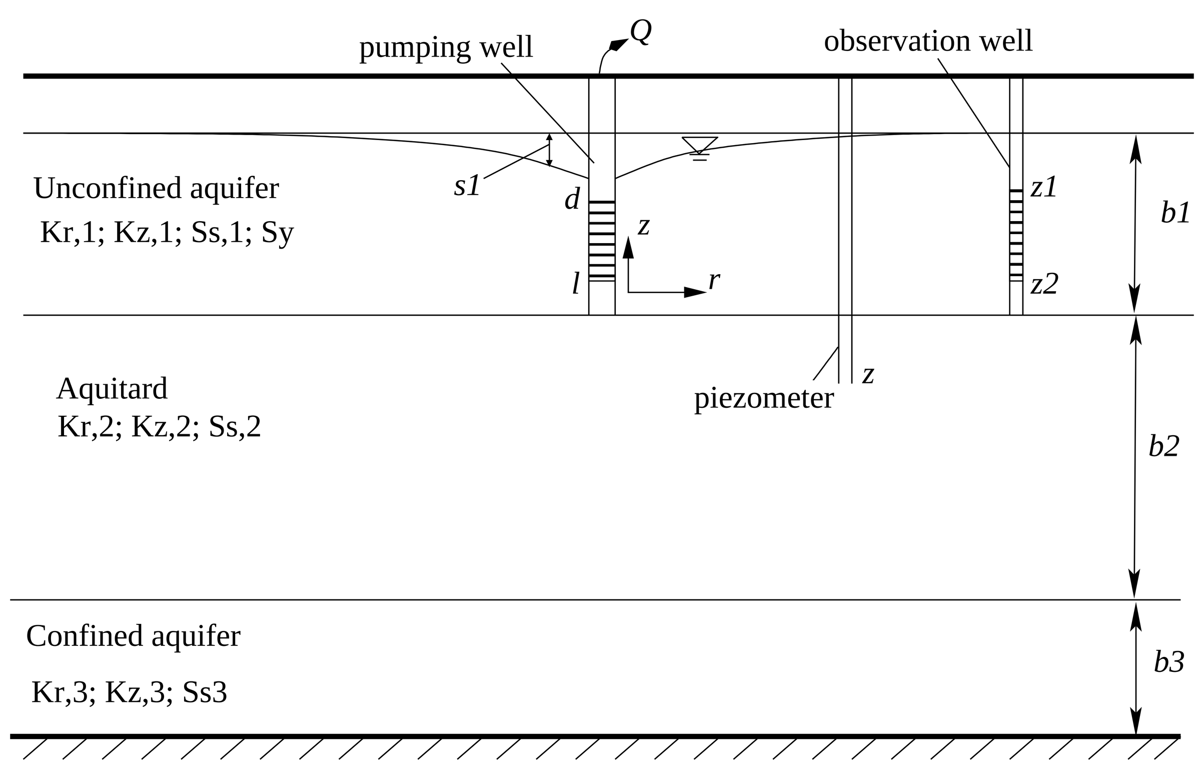

Figure 1: Schematic of a 3-layered unconfined-aquitard-confined aquifer system.

Solution in Laplace-Hankel transform space

$$\overline{s}_{D,1}^*(a, z_D, p) = \overline{u}_D^*(a, z_D, p) + \overline{v}_D^*(a, z_D, p) \tag{11}$$

where

$$\overline{u}_D^*(a, z_D, p) = \frac{2\left[\cosh(\eta_1\zeta_D) - \delta\overline{u}_D^*(\eta_1, z_D)\right]}{p\eta_1^2\alpha_{D,z}^{(1)}(l_D - d_D)}, \tag{12}$$

$$\zeta_D = \begin{cases} d_D + z_D & \forall z_D \in (-d_D, 0] \\ 0 & \forall z_D \in [-l_D, -d_D], \\ l_D + z_D & \forall z_D \in [-1, -l_D) \end{cases} \tag{13}$$

$$\delta\overline{u}_D^*(\eta_1, z_D) = \frac{\sinh(\eta_1 d_D)\cosh\left[\eta_1(1+z_D)\right] + \sinh\left[\eta_1(1-l_D)\right]\cosh(\eta_1 z_D)}{\sinh(\eta_1)}, \tag{14}$$

and

$$\overline{v}_D^*(a, z_D, p) = \overline{u}_{D,-1}^*\cosh\left[\eta_1(1+z_D)\right] + \frac{f(\eta_1)h(a, z_D, p)}{\Delta} \tag{15}$$

with

$$f(\eta_1) = \overline{u}_{D,0}^* - \overline{u}_{D,-1}^*\left[\xi\sinh(\eta_1) + \cosh(\eta_1)\right], \tag{16}$$

$$\begin{aligned} h(a, z_D, p) = {} & \frac{(1+\gamma_1)(1+\gamma_2)}{2}\cosh\left[\eta_1(1+z_D)+\theta_1\right] + \frac{(1-\gamma_1)(1+\gamma_2)}{2}\cosh\left[\eta_1(1+z_D)-\theta_1\right] \\ & + \frac{(1-\gamma_1)(1-\gamma_2)}{2}\cosh\left[\eta_1(1+z_D)+\theta_2\right] + \frac{(1+\gamma_1)(1-\gamma_2)}{2}\cosh\left[\eta_1(1+z_D)-\theta_2\right] \end{aligned} \tag{17}$$

$$\begin{aligned} \Delta = {} & \frac{(1+\gamma_1)(1+\gamma_2)}{2}\left[\xi\sinh(\eta_1+\theta_1) + \cosh(\eta_1+\theta_1)\right] \\ & + \frac{(1-\gamma_1)(1+\gamma_2)}{2}\left[\xi\sinh(\eta_1-\theta_1) + \cosh(\eta_1-\theta_1)\right] \\ & + \frac{(1-\gamma_1)(1-\gamma_2)}{2}\left[\xi\sinh(\eta_1+\theta_2) + \cosh(\eta_1-\theta_2)\right] \\ & + \frac{(1+\gamma_1)(1-\gamma_2)}{2}\left[\xi\sinh(\eta_1-\theta_2) + \cosh(\eta_1+\theta_2)\right] \end{aligned} \tag{18}$$

$\xi = \eta_1\alpha_{D,Y}/p$, $\gamma_1 = \eta_2 K_{D,z}^{(2)}/\eta_1$, $\gamma_2 = \eta_3 K_{D,z}^{(3)}/\eta_2$, $\theta_1 = \eta_3(b_{D,3} - b_{D,2}) + \eta_2(b_{D,2} - 1)$, $\theta_2 = \eta_3(b_{D,3} - b_{D,2}) - \eta_2(b_{D,2} - 1)$

Numerical inversion of Laplace-Hankel transform

Inverse Laplace and Hankel transforms are obtained numerically.

i. The inverse Laplace transform was obtained using the method of (DHKS82), and;

ii. The inverse Hankel transform was obtained in a manner similar to that proposed by (Wie99)

Results

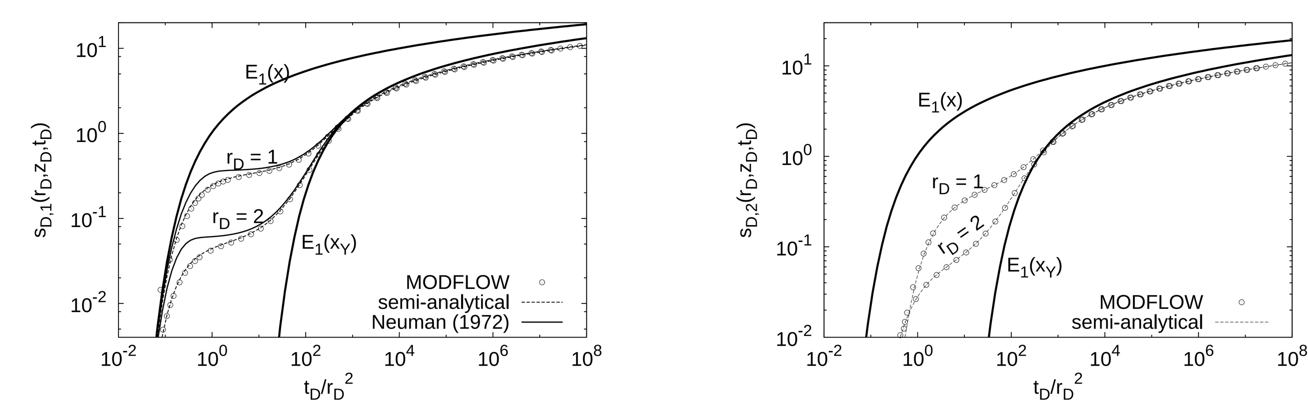

Figure 2: Comparison of $s_{D,1}$ (left) and $s_{D,2}$ (right) with numerical solution obtained with MODFLOW-2000.

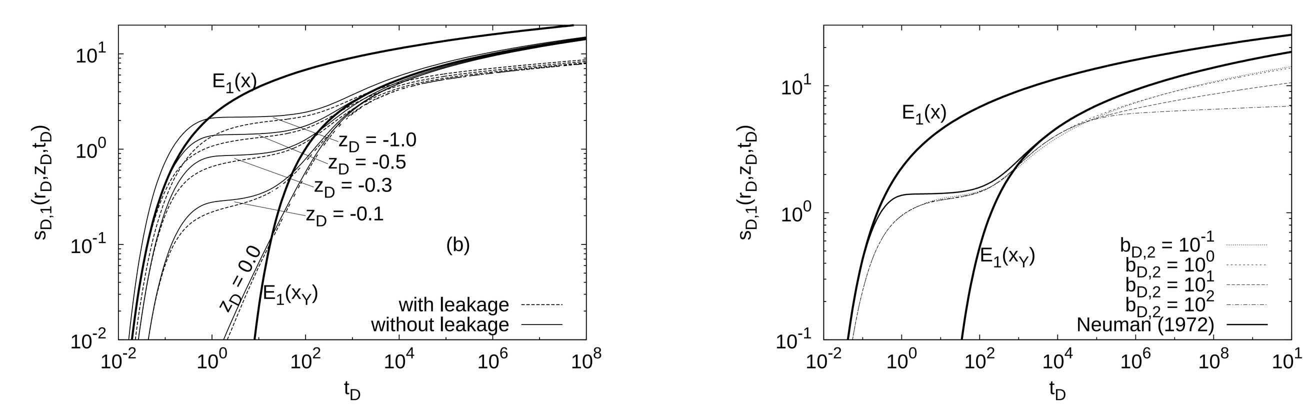

Figure 3: Log-log plot of point values of $s_{D,1}$ vs. t_D for different values of z_D (left) and $b_{D,2}$ (right).

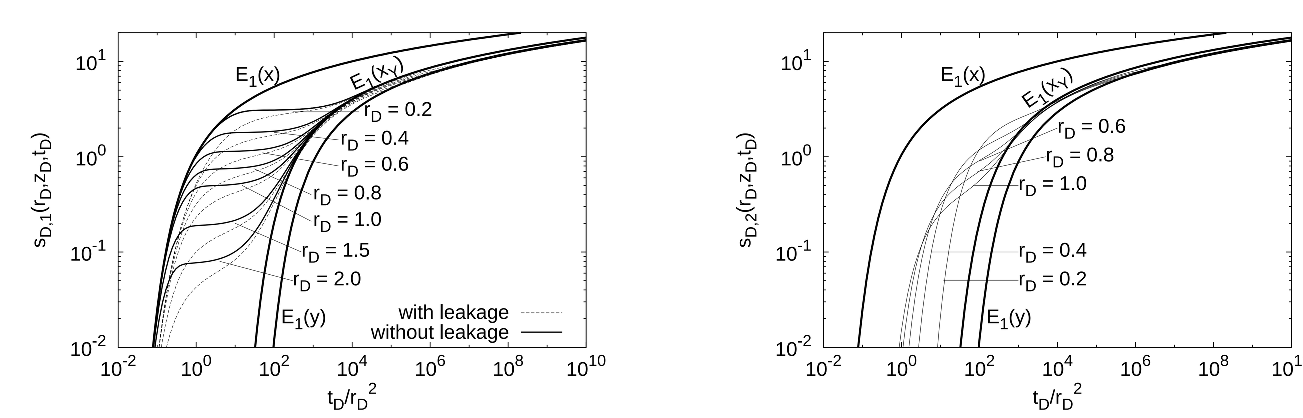

Figure 4: Log-log plot of point values of $s_{D,1}$ vs. t_D (left) and $s_{D,2}$ vs. t_D (right) for different r_D.

Summary

i. Leakage leads to significant departure from behavior predicted by solution for no leakage

ii. Early time departure due to leakage from aquitard storage

iii. Late time departure due to conveyance of fluid from the far reaches of aquitard and confined aquifer to pumping well

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