Advances in LT-AEM Theory and Application to Transient Groundwater Flow

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Introduction

The LT-AEM — like the AEM developed by Otto Strack — falls somewhere in between gridded methods (*e.g.*, finite element (FEM)) and fully analytic solutions (*e.g.*, Theis solution). It can be thought of as a flexible *quasi-analytic* solution which can be easily adjusted for different conditions or situations; the LT-AEM is similar in philosophy to the boundary element method (BEM).

The Laplace transform is used to remove the time derivative from the governing groundwater flow equation (the diffusion equation), giving the modified Helmholtz equation in the Laplace domain:

$$\underbrace{\mathcal{L}\left[\nabla^2\Phi = \frac{1}{\alpha}\frac{\partial\Phi}{\partial t}\right]}_{\text{diffusion (transient)}} \longrightarrow \underbrace{\nabla^2\bar{\Phi} = \frac{1}{\alpha}\left(\bar{\Phi}p - \Phi_0\right)}_{\text{modified Helmholtz (steady-state)}} \qquad (1)$$

where Φ is discharge potential ($\Phi = Kh$), α is hydraulic diffusivity ($\alpha = K/S_s$), Φ_0 is the initial condition (set to zero to make the equation homogeneous), p is the Laplace parameter, and the over-bar indicates a Laplace transformed variable.

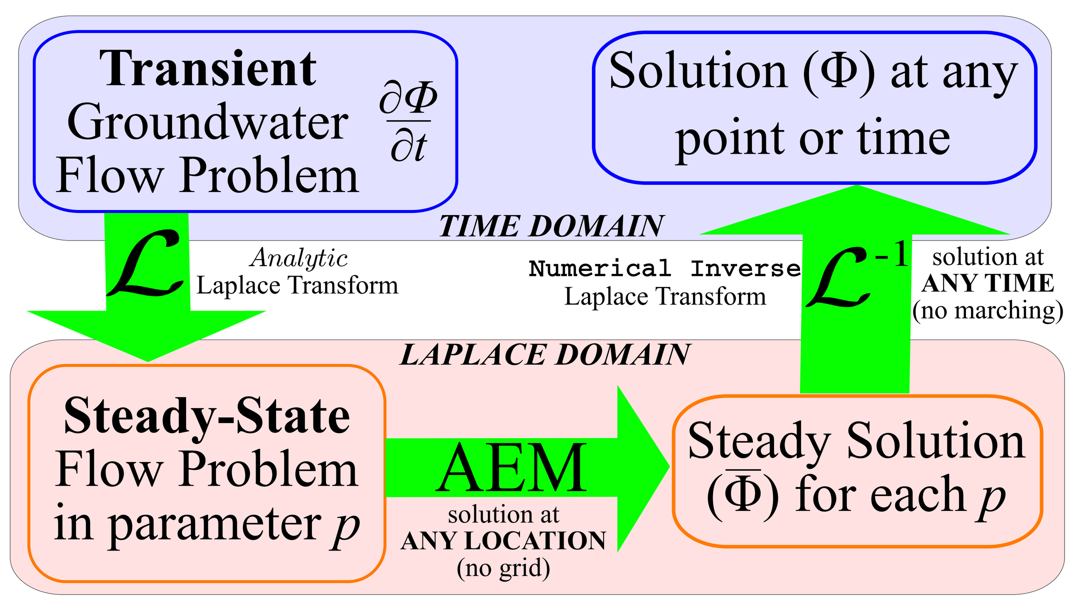


FIG. 1: Laplace Transform Analytic Element Method (LT-AEM) flowchart

The time solution is estimated with several AEM solutions, for different values of p, using a numerical inverse Laplace transform algorithm to approximate the Bromwich contour integral. Since p is a *parameter*, these calculations can be carried in any order or on any processor — making parallel computations easy to implement.

LT-AEM elements can be active or passive. Passive elements have their strength specified independent of the other elements in the problem; active elements depend on the behavior of elements throughout the domain.

LT-AEM process

1. **Define transient elements** which compose problem:
   e.g., wells → points, rivers → lines, areas of recharge → circles or ellipses;
2. Pose problem in **Laplace space**: $\mathcal{L}\left[\Phi(t)\right] = \bar{\Phi}(p)$;
3. Calculate each **passive element's** effect explicitly:
   e.g., drawdown due to pumping well at constant Q;
4. Calculate each **active element's** strength iteratively:
   e.g., a line source which keeps head at $sin(t)$;
5. Use **superposition in Laplace space** to find total solution,
   $\bar{\Phi}_{total}(p) = \sum\bar{\Phi}_{ellipse}(p) + \sum\bar{\Phi}_{circle}(p) + \sum\bar{\Phi}_{line}(p) + \ldots$;
6. **Numerically invert** total solution: $\mathcal{L}^{-1}\left[\bar{\Phi}(p)\right] = \Phi(t)$;

Elliptical Elements

Furman and Neuman developed the LT-AEM point and circle elements as a proof of concept of the method [1]. LT-AEM elements use a collocation approach to match conditions approximately at element boundaries, while the elements themselves are chosen to satisfy the PDE exactly throughout the domain (like in BEM).

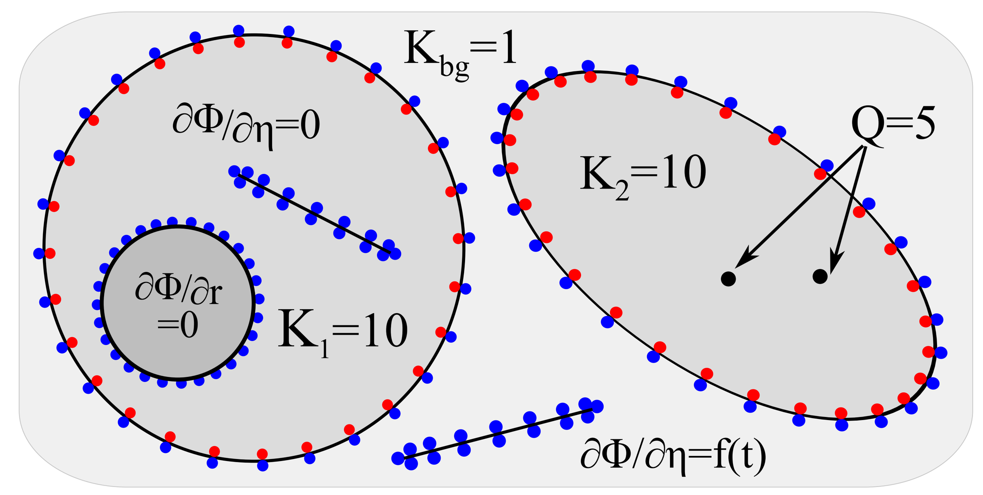
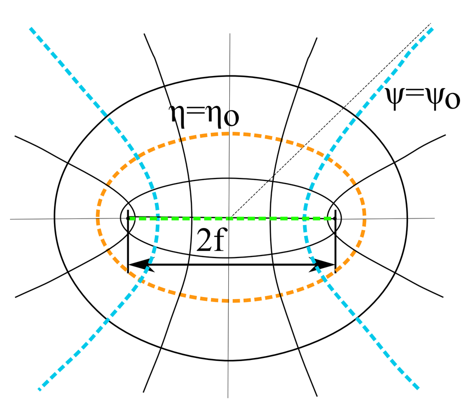


FIG. 2: Matching (collocating) points along elements can be used to match h and $\partial\Phi/\partial n$ between regions of different properties or specify arbitrary boundary conditions.

Using separation of variables, the domains inside and outside an element are represented by the natural eigenfunctions for that shape. Ellipses and finite line segments (which can be considered degenerate ellipses) are natural in elliptical coordinates (see Figure at right). In these coordinates the Helmholtz equation has modified radial and angular Mathieu functions as eigenfunctions.



For example, to represent the effect of an elliptical recharge area having different hydraulic conductivity, the total head and total normal flux are matched along the boundary of the ellipse. The total discharge potential, $\bar{\Phi}^{\pm}_{\text{total}} = \bar{\Phi}^{\pm}_{\text{ellipse}} + \sum\bar{\Phi}^{\pm}_{\text{background}}$, is the sum of the effects of the current element and all the elements in the background (+ outside and − inside the ellipse). The following matching equations are posed at points along the boundary of the elements (blue and red points in Figure 2):

$$\overbrace{\left[\frac{\bar{\Phi}^{+}_{\text{total}}}{K^{+}}\right]_{\eta_0} = \left[\frac{\bar{\Phi}^{-}_{\text{total}} + \bar{\Phi}^{-}_{\text{part}}}{K^{-}}\right]_{\eta_0}}^{\text{head matching}} \qquad \overbrace{\left[\frac{\partial\bar{\Phi}^{+}_{\text{total}}}{\partial\eta}\right]_{\eta_0} = \left[\frac{\partial\left(\bar{\Phi}^{-}_{\text{total}} + \bar{\Phi}^{-}_{\text{part}}\right)}{\partial\eta}\right]_{\eta_0}}^{\text{normal flux matching}}, \qquad (2)$$

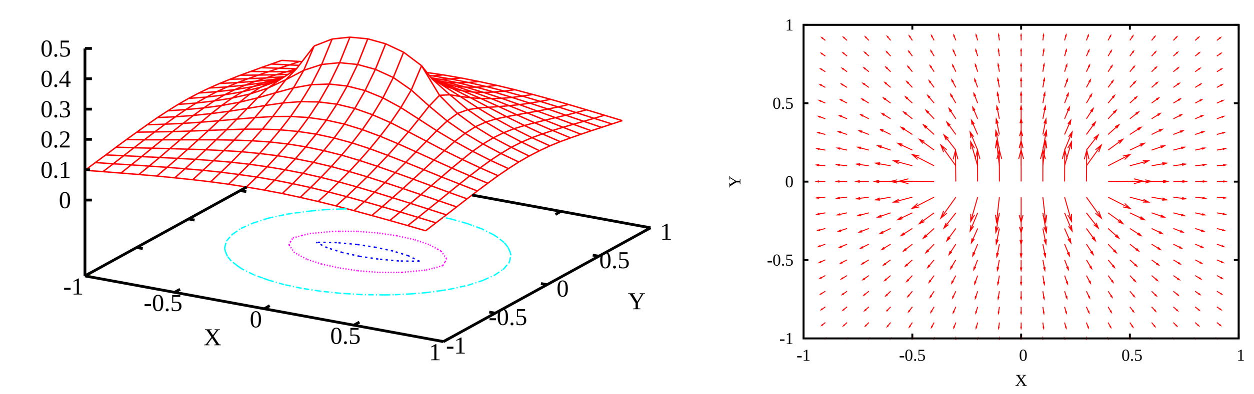
where η_0 is the boundary of the ellipse and $\bar{\Phi}^{-}_{\text{part}}$ is the particular solution associated with the areal recharge flux inside the element (the non-homogeneous term in 1). The following expressions for the current ellipse are obtained through separation of variables,

$$\bar{\Phi}^{+}_{\text{ellipse}}(\eta \geq \eta_0, \psi) = \sum_{\nu=0}^{\infty} a_\nu \ \text{ce}_\nu(q,\psi)\ \text{Ke}_\nu(q,\eta) + \sum_{\nu=1}^{\infty} b_\nu \ \text{se}_\nu(q,\psi)\ \text{Ko}_\nu(q,\eta),$$

$$\bar{\Phi}^{-}_{\text{ellipse}}(\eta \leq \eta_0, \psi) = \sum_{\nu=0}^{\infty} c_\nu \ \text{ce}_\nu(q,\psi)\ \text{Ie}_\nu(q,\eta) + \sum_{\nu=1}^{\infty} d_\nu \ \text{se}_\nu(q,\psi)\ \text{Io}_\nu(q,\eta), \qquad (3)$$

where the functions ce and se are analogous to sin and cos, while Ie, Io, Ke and Ko are even and odd analogs to the modified Bessel functions, I and K. The parameter, $q = (f^2p)/(2\alpha)$, is related to aquifer properties and the shape of the ellipse.

Substituting the expressions for $\Phi^{\pm}_{\text{ellipse}}$ and their derivatives into (2) allows estimating the unknown coefficients a_ν, b_ν, c_ν and d_ν using least-squares, by posing the matching expressions (2) at more points than there are unknowns in (3). If the background elements, $\bar{\Phi}^{\pm}_{\text{background}}$, are also active, estimating the parameters of each becomes a non-linear problem (each active element depending on the strength of every other element), which can be solved quickly using fixed-point iteration.



constant strength line source head (h) and seepage velocity $\left(\frac{1}{n}\nabla\Phi\right)$

These figures illustrate a line source with strength constant in space and time, simulated using elliptical elements ($\eta_0 = 0, f = 0.35$). The benefit of representing lines as ellipses is generality, nearly any variation in space or time can be represented.

LT-AEM Applications

Since an LT-AEM simulation for simple flow problems is fast and accurate, the approach makes a good learning tool. Cutting a homogeneous rectangular domain into hundreds of square elements to see the response from a line or point source (as would be done with MODFLOW) isn't really an intuitive way to visualize what is going on. Instructional use, "what-if" planning — moving wells around in a wellfield to examine their cumulative effects — or the testing phase before a larger gridded modeling project would all capitalize on strengths of the LT-AEM.

LT-AEM models could also be coupled with FDM or FEM models to utilize the benefits of each method. Since the LT-AEM deals with "infinite" domains or distant boundaries quite naturally (without hundreds of intervening elements), a study-area gridded model could be surrounded by an LT-AEM model (like grid telescoping); the LT-AEM would provide boundary conditions for the study-area model.

Future Work & Support

The LT-AEM is already implemented as a working Fortran95 program; therefore the current effort is development of new LT-AEM elements, which can be used to build more general transient groundwater flow models. The final product of this research will be a general groundwater modeling environment with a friendly graphical interface (such as that already developed by our USGS collaborator, Paul Hsieh, in Menlo Park, CA).

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References

[1] A. Furman and S. P. Neuman. Laplace-transform analytic element solution of transient flow in porous media. *Advances in Water Resources*, 26:1229–1237, 2003.