

Models for Unsaturated Hydraulic Conductivity Based on Truncated Lognormal Pore-size Distributions

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Abstract

We develop a closed-form three-parameter model for unsaturated hydraulic conductivity associated with a three-parameter lognormal model of moisture retention, which is based on lognormal grainsize distribution. The derivation of the model is made possible by a slight modification to the theory of Mualem. We extend the three-parameter lognormal distribution to a four-parameter model that also truncates the pore size distribution at a minimum pore radius. We then develop the corresponding four-parameter model for moisture retention and the associated closed-form expression for unsaturated hydraulic conductivity. The four-parameter model is fitted to experimental data, similar to the models of Kosugi and van Genuchten. The proposed four-parameter model retains the physical basis of Kosugi's model, while improving fit to observed data especially when simultaneously fitting pressure-saturation and pressure-conductivity data.

Keywords: lognormal distribution, moisture retention model, unsaturated hydraulic conductivity

1. Introduction

Kosugi (1994) assumed pore size is a lognormal random variable and derived a three-parameter model for moisture retention, the three parameters being the mean and variance of the pore-size distribution and the maximum pore radius. In the limiting case where the maximum pore radius approaches infinity the three-parameter model simplifies to a two-parameter model for which Kosugi (1996) developed the closed-form expression for unsaturated hydraulic conductivity using the theory of Mualem (1976). Kosugi (1996) did not develop a three-parameter closed-form equation for hydraulic conductivity, but reverted to the two-parameter form, owing to difficulty in analytically integrating the expression of Mualem (1976). We extend the work of Kosugi (1994, 1996) and develop closed-form expressions for unsaturated hydraulic conductivity associated with the three-parameter lognormal moisture retention model. The derivation of the closed-form equation for unsaturated hydraulic conductivity is made possible by a slight modification to the theory of Mualem (1976). Further, we modify the pore-size probability density function (PDF) of Kosugi (1994) by incorporating a nonzero minimum pore radius, as suggested by Brutsaert (1966). This modification results

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14 in a four-parameter moisture retention model and a corresponding four-parameter closed-form equation for
 15 unsaturated hydraulic conductivity, again obtained using the modified theory of Mualem (1976). The four
 16 parameters in the proposed model are all based on physical properties of the porous medium (similar to the
 17 model of Kosugi (1996)), not fitting parameters without physical significance.

18 2. Theory

19 The lognormal distribution is commonly used to statistically characterize pore size in granular porous
 20 media. Brutsaert (1966) and Kosugi (1994, 1996) considered lower- and upper-tail truncated lognormal PDFs
 21 for pore-size distributions. Brutsaert (1966) considered the log-transformed random pore radius $R-r_0$, where
 22 r_0 is the radius at which the effective moisture content vanishes (associated with residual saturation).

23 2.1. The three-parameter lognormal model

24 The classical (non-truncated) lognormal distribution for pore size (Brutsaert, 1966), measured here by the
 25 random pore radius $R \in [0, \infty]$. For a real porous medium $R \in [0, r_{\max}]$, where r_{\max} is some finite maximum
 26 pore radius. To correct for the finite interval, Kosugi (1994) used the random variable $R_e = (1/R - 1/r_{\max})^{-1}$
 27 to rescale the classical PDF. The PDF of R is related to the PDF of R_e according to

$$28 \quad f_R(r) = \frac{f_{R_e} \left[(1/r - 1/r_{\max})^{-1} \right]}{(1 - r/r_{\max})^2}. \quad (1)$$

29 According to Young-Laplace theory, the capillary pressure head h and pore radius r are related according
 30 to $h = \kappa/r$, where $\kappa = 2\gamma \cos \alpha / (\rho g)$, γ is interface surface tension, α is the interface contact angle, ρ is fluid
 31 density, and g is gravitational acceleration. For water in a glass tube, $\kappa \approx 0.149 \text{ cm}^2$. It then follows that
 32 the PDF of the random capillary pressure head H is

$$33 \quad f_H(h) = \frac{f_{R_e} \left[(h/\kappa - 1/r_{\max})^{-1} \right]}{(h/\kappa - 1/r_{\max})^2}. \quad (2)$$

34 Further, if R_e is lognormally distributed, the PDF for the capillary pressure head H can be written as

$$35 \quad f_H(h) = \frac{1}{\sqrt{2\pi}\sigma_Z(h - h_c)} \exp \left[- \left(\frac{\log(h - h_c) - \mu_\eta}{\sqrt{2}\sigma_Z} \right)^2 \right], \quad (3)$$

36 for all $h > h_c$, where $h_c = \kappa/r_{\max}$ is the bubbling pressure head, $\mu_\eta = \log(\kappa) - \mu_Z$ is the mean of $\log(H)$, σ_Z^2
 37 is the variance of Z , μ_Z is the mean of Z , and $Z = \log(R_e)$. Kosugi (1994) used the dimensionless random
 38 variable $R'_e = R_e/r_{\max}$ to obtain the PDF in (3) with the parameters μ_Z and σ_Z^2 scaled appropriately.

39 As shown by Kosugi (1994), the three-parameter moisture retention curve that follows from (3) is

$$40 \quad \theta^*(h) = \begin{cases} \frac{1}{2} \operatorname{erfc} \left(\frac{\log(h - h_c) - \mu_\eta}{\sqrt{2}\sigma_Z} \right) & h > h_c, \\ 1 & h \leq h_c, \end{cases} \quad (4)$$

41 where $\theta^*(h) = (\theta(h) - \theta_r)/(\theta_s - \theta_r)$ is moisture capacity, $\theta(h)$ is volumetric moisture content, θ_r is residual
 42 moisture content, θ_s is saturated moisture content, and erfc is the complementary error function. Kosugi
 43 (1994) did not develop a corresponding closed-form equation for unsaturated hydraulic conductivity.

44 Mualem (1976) developed a functional relation between unsaturated hydraulic conductivity $K(\theta^*) =$
 45 $K_s K_r$ and capillary pressure head h

$$46 \quad K_r(\theta^*) = \sqrt{\theta^*} \left[\left(\int_0^{\theta^*} \frac{dx}{h(x)} \right) / \left(\int_0^1 \frac{dx}{h(x)} \right) \right]^2, \quad (5)$$

47 where K_r and K_s are relative and saturated hydraulic conductivity and x is an integration variable. Equation
 48 (5) can be rewritten in terms of the PDF of capillary pressure head as

$$49 \quad K_r(\theta^*) = \sqrt{\theta^*} \left[\left(\int_h^\infty \frac{f_H(x)}{x} dx \right) / \left(\int_0^\infty \frac{f_H(x)}{x} dx \right) \right]^2. \quad (6)$$

50 Using the theory of Mualem (1976), Kosugi (1996) developed the two-parameter closed-form equation for
 51 unsaturated hydraulic conductivity by forcing $r_{\max} \rightarrow \infty$. Kosugi (1996) made this simplification because
 52 the theory of Mualem (1976) as given in (6) is not readily amenable to integration when r_{\max} in the three-
 53 parameter lognormal distribution is finite.

54 We obtain a closed-form expression for unsaturated hydraulic conductivity for finite values of r_{\max} by
 55 modifying (5) due to Mualem (1976) into

$$56 \quad K_r(\theta^*) = \sqrt{\theta^*} \left[\left(\int_0^{\theta^*} \frac{dx}{h(x) - h_c} \right) / \left(\int_0^1 \frac{dx}{h(x) - h_c} \right) \right]^2, \quad (7)$$

57 based on the assumption $f_H(h)/h \approx f_H(h)/(h - h_c)$. Using this approximation the closed-form unsaturated
 58 hydraulic conductivity expression for the three-parameter lognormal model is

$$59 \quad K_r(h) = \begin{cases} \sqrt{\theta^*} \left\{ \frac{1}{2} \text{erfc} \left[\frac{\log(h-h_c) - \mu_\eta + \sigma_z^2}{\sqrt{2}\sigma_z} \right] \right\}^2 & h > h_c, \\ 1 & h \leq h_c. \end{cases} \quad (8)$$

60 In the limit as $r_{\max} \rightarrow \infty$, (8) reduces to the two-parameter unsaturated hydraulic conductivity expression
 61 derived by Kosugi (1996). Figure 1 shows that the truncated lognormal pore-size distribution (8) results in
 62 the conductivity curves shifted to the right ($K_r = 1$ at $h = h_c$) compared to the curves corresponding to the
 63 two-parameter model of Kosugi (1996), where $K_r = 1$ is reached at $h > 0$.

64 2.2. Four-parameter lognormal model

65 To incorporate the lower-tail truncation of Brutsaert (1966) into the distribution of Kosugi (1994), we
 66 introduce the random variable \hat{R}_e defined by

$$67 \quad \hat{R}_e = \left(\frac{1}{R - r_0} - \frac{1}{r_{\max}} \right)^{-1}, \quad (9)$$

68 which can be shown to yield the following lognormal PDF for capillary pressure head,

$$69 \quad f_H(h) = \frac{1}{\sqrt{2\pi}u\sigma_Z} \exp \left[- \left(\frac{\log(u) - \mu_\eta}{\sqrt{2}\sigma_Z} \right)^2 \right], \quad (10)$$

70 for all $h \in [h_c, h_{\max}]$ where $u = (1/h - 1/h_{\max})^{-1} - h_c$ and $h_{\max} = \kappa/r_0$ is the pressure head associated with
 71 the smallest undrainable pores in the medium corresponding to the physical boundary on pore radius. Figure
 72 2 shows the three lognormal PDFs for capillary pressure head: the classical (non-truncated) lognormal dis-
 73 tribution, the upper-truncated lognormal distribution (3), and the doubly truncated lognormal distribution
 74 (10). It can be seen in the figure the model of Kosugi (1994) departs from the classical lognormal distribution
 75 only at small head values (large pore radii) whereas the proposed four-parameter distribution departs from
 76 the classical distribution at both the lower and upper limbs of the function.

77 The moisture retention curve is derived from (10) in a manner similar to that done by Kosugi (1996) for
 78 the four-parameter lognormal distribution, and is given by

$$79 \quad \theta^*(h) = \begin{cases} \frac{1}{2} \operatorname{erfc} \left[\frac{\log(u) - \mu_\eta}{\sqrt{2}\sigma_Z} \right] & h_c < h < h_{\max}, \\ 1 & h \leq h_c, \\ 0 & h \geq h_{\max}. \end{cases} \quad (11)$$

80 Finally, it can be shown that the closed-form expression for unsaturated hydraulic conductivity using the
 81 doubly truncated PDF (10) and the modified equation of Mualem (1976) (6) is

$$82 \quad K_r(h) \simeq \begin{cases} \sqrt{\theta^*} \left\{ \frac{1}{2} \operatorname{erfc} \left[\frac{\log(u) - \mu_\eta - \sigma_Z^2}{\sqrt{2}\sigma_Z} \right] \right\}^2 & h_c < h < h_{\max}, \\ 1 & h \leq h_c, \\ 0 & h \geq h_{\max}. \end{cases} \quad (12)$$

83 In the limit as both $r_{\max} \rightarrow \infty$ and $r_0 \rightarrow 0$, (11) and (12) reduce to corresponding two-parameter expressions
 84 from Kosugi (1994, 1996).

85 3. Fitting four-parameter lognormal model to experimental data

86 The four-parameter lognormal model for moisture retention (11) was fitted to experimental data (same
 87 data used by van Genuchten (1980) and Kosugi (1996)) using non-linear least squares to estimate the
 88 parameters r_0 , r_{\max} , μ_Z and σ_Z^2 . We also estimated θ_r and θ_s from experimental data, rather than using
 89 reported values. Using estimated parameters in the four-parameter model (12), predictions of unsaturated
 90 hydraulic conductivity were compared to measured values. The results for Hygiene sandstone and silt loam
 91 G.E. 3 are shown in Figures 3 and 4. A summary of estimated parameters for these soils is given in Table 1.
 92 Figure 3(b) shows a solid curve computed using the approximate, but closed-form expression for K_r (7) and a
 93 dashed curve numerically integrated from the traditional Mualem relationship (5), which are nearly identical
 94 for this set of parameters. The fits to experimental data are comparable to those of Kosugi (1996) and

95 van Genuchten (1980). Markov-chain Monte Carlo simulations with the four-parameter model and moisture
96 retention curve data show that the logarithms of parameters r_0 and r_{\max} are sometimes identifiable only in
97 a threshold sense. No well-defined optimal value exists, the data imply the log minimum pore size is less
98 than a threshold, or the log maximum is greater than a threshold.

99 The four-parameter lognormal model was also fitted to moisture retention data for Beit Netofa clay, using
100 these estimated parameter values to make predictions of conductivity. The four-parameter model yielded
101 predictions of unsaturated conductivity comparable to the two-parameter models of Kosugi (1996) and van
102 Genuchten (1980) as shown by the dotted curves in Figure 5(a) and (b) (labeled 2- and 3-parameter). How-
103 ever, when hydraulic conductivity and moisture retention data were jointly used to estimate the parameters
104 r_0 , r_{\max} , μ_Z , σ_Z^2 , θ_r , and θ_s , there was marked improvement in model fits to the data as shown by the
105 solid curves in Figure 5(a) and (b) (labeled 4-parameter). The parameter values used in the fitted model
106 for moisture retention data shown in Figure 5(a) are the same values used for the fitted conductivity model
107 in Figure 5(b). The same procedure did not yield similar results when applied to the the two-parameter
108 model of Kosugi (1996), as shown by the dashed curves in plots (a) and (b) of Figure 5, which were obtained
109 by jointly using moisture retention and conductivity data to estimating the parameters μ_Z and σ_Z^2 , with
110 $r_{\max} \rightarrow \infty$.

111 4. Discussion

112 The proposed four-parameter lognormal model fits moisture retention data from three representative
113 soils similar to the models of van Genuchten (1980) and Kosugi (1996). Predictions of unsaturated hydraulic
114 conductivity are comparable to those of van Genuchten (1980) and Kosugi (1996) for Hygiene sandstone and
115 silt loam G.E. 3. Estimating model parameters for Beit Netofa clay using only moisture retention data does
116 not yield good predictions of unsaturated hydraulic conductivity (similar to the models of van Genuchten
117 (1980) and Kosugi (1996)). It was essential to use both moisture retention and hydraulic conductivity data
118 to estimate the parameters and improve the fit of the proposed model over the models of Kosugi (1994) and
119 van Genuchten (1980). This indicates moisture retention data alone may not be sufficient to estimate all
120 four (or six, when including θ_r and θ_s) model parameters. If conductivity measurements are available, they
121 should be used with moisture retention data, to arrive at a more realistic closed-form model for unsaturated
122 soil moisture retention and hydraulic conductivity.

123 One would expect a model with a larger number of adjustable parameters to fit observed data better than
124 similar models with less parameters, but an improved fit is often at the expense of the physical significance of
125 the parameters. The proposed four-parameter model does not make its gains in model-data fit at the expense
126 of parameter realism. The parameters follow the philosophy Kosugi (1994) used in deriving his model: they
127 are related to pore-size distribution statistics, rather than being fitting parameters. The development and use
128 of the modified model of Mualem (1976) enables one to obtain approximate but closed-form expressions for

129 unsaturated hydraulic conductivity for both the three- and four-parameter lognormal pore size distributions.
130 The model derived here is a generalization of the the lognormal model of Kosugi (1996) for non-zero minimum
131 pore radius and non-infinite maximum pore radius.

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144 soils. *The Soil Science Society of America Journal* 44 (5), 892–898.

Table 1: Estimated parameter values for proposed 4-parameter model

| Medium | $r_0(\text{m})$ | $r_{\max}(\text{m})$ | μ_Z | σ_Z | θ_s | θ_r |
|-------------------|-----------------------|-----------------------|---------|------------|------------|------------|
| Hygiene Sandstone | 1.07×10^{-4} | 2.52×10^{-3} | -6.300 | 0.337 | 0.250 | 0.153 |
| Silt Loam G.E. 3 | 1.48×10^{-4} | 1.27×10^{-2} | -7.927 | 1.118 | 0.395 | 0.192 |
| Beit Netofa Clay | 4.36×10^{-7} | 1.32×10^{-2} | -11.06 | 2.333 | 0.450 | 0.100 |

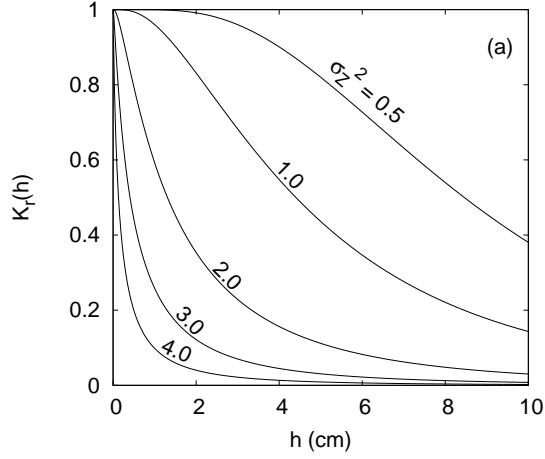


Figure 1: Predicted $K_r(h)$ for (a) the two-parameter model of Kosugi (1996), and (b) the proposed three-parameter lognormal model for finite r_{\max} .

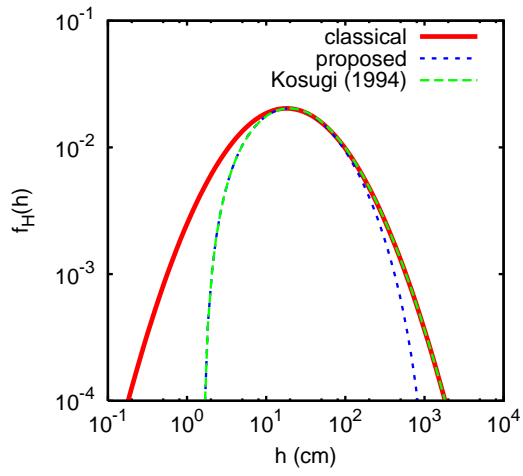


Figure 2: Comparison of lognormal capillary pressure head PDFs for the classical, three-parameter (Kosugi, 1994), and proposed four-parameter (10) models.

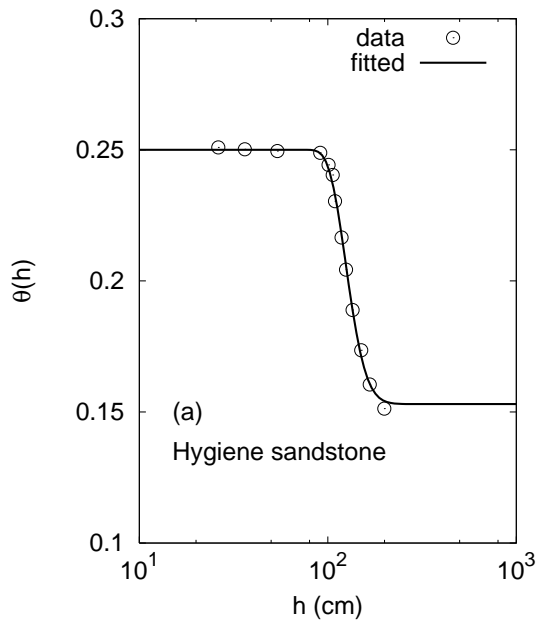


Figure 3: (a) Moisture content data and fitted moisture retention curve, and (b) measured and predicted K_r (using both analytical and numerical integration) for Hygiene Sandstone (Mualem (1976) soil index 4130).

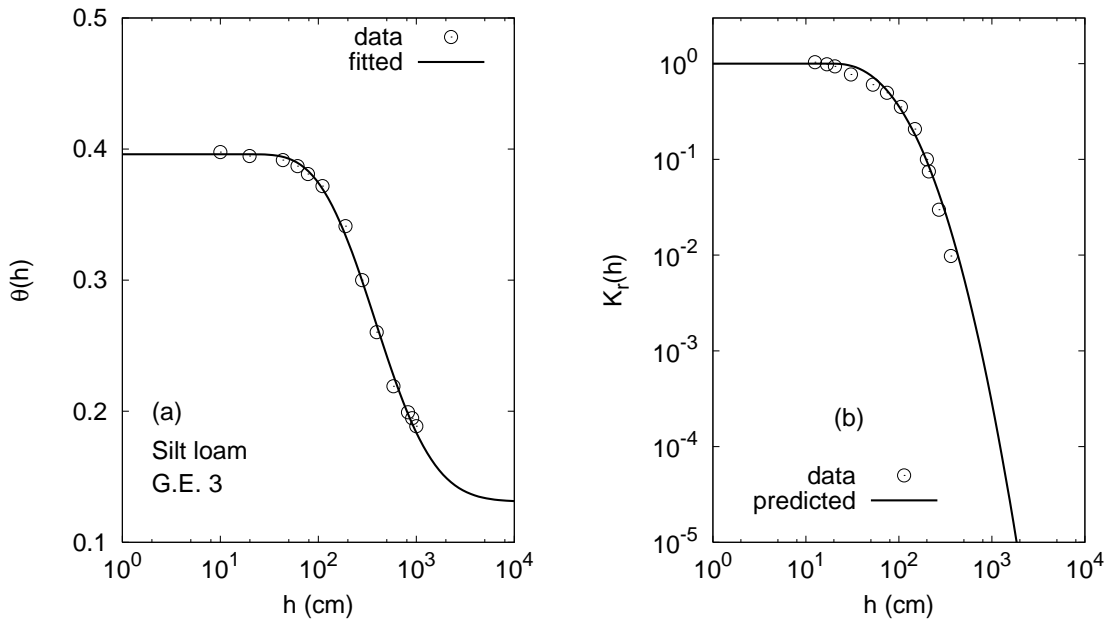


Figure 4: (a) Moisture content data and fitted moisture retention curve, and (b) measured and predicted K_r for Silt Loam G.E. 3 (Mualem (1976) soil index 3310).

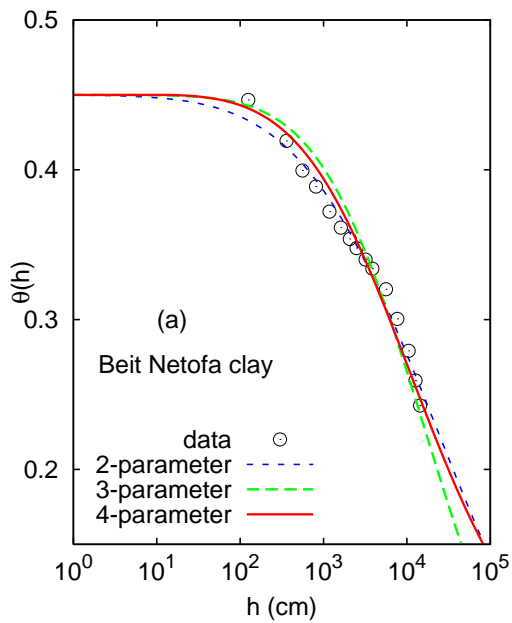


Figure 5: (a) Moisture content data and fitted moisture retention curve, and (b) measured and fitted K_r for Beit Netofa Clay (Mualem (1976)) soil index 1006).