Unconfined Aquifer Flow Theory - from Dupuit to present

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1.1 Introduction

Pumping tests are widely used to obtain estimates of hydraulic parameters characterizing flow and transport processes in subsurface (e.g., Kruseman and de Ridder (1990); Batu (1998)). Hydraulic parameter estimates are often used in planning or engineering applications to predict flow and design of aquifer extraction or recharge systems. During a typical pumping test in a horizontally extensive aquifer, a well is pumped at constant volumetric flow rate and head observations are made through time at one or more locations. Pumping test data are presented as time-drawdown or distance-drawdown curves, which are fitted to idealized models to estimate hydraulic properties of the aquifer. For unconfined aquifers properties of interest include hydraulic conductivity, specific storage, specific yield, and possibly unsaturated flow parameters. When estimating aquifer properties using pumping test drawdown data, one can use a variety of analytical solutions involving different conceptualizations and simplifiying assumptions. Analytical solutions are impacted by their simplifiying assumptions, which limit their applicability to characterize certain types of unconfined aquifers. This review presents the historical evolution of the scientific and engineering thoughts concerning groundwater flow towards a pumping well in unconfined aquifers (also referred to variously as gravity, phreatic, or water table aquifers) from the steady-state solutions of Dupuit to the coupled transient saturated-unsaturated solutions of the present. Although simulation using gridded numerical models is sometimes necessary in highly irregular or heterogeneous systems, here we limit our consideration to analytically derived solutions.

1.2 Early Well Test Solutions

1.2.1 Dupuit's Steady-State Finite-Domain Solutions

Dupuit (1857) considered steady-state radial flow to a well pumping at constant volumetric flowrate Q [L^3/T] in a horizontal homogeneous confined aquifer of thickness b [L]. He used Darcy's law (Darcy, 1856) to express the velocity of groundwater flow u [L/T] in terms of radial hydraulic head gradient $(\partial h/\partial r)$ as

$$u = K\frac{\partial h}{\partial r}, \tag{1.1}$$

where $K = kg/\nu$ is hydraulic conductivity [L/T], k is formation permeability [L^2], g is the gravitational constant [L/T^2], ν is fluid kinematic viscosity [L^2/T], $h = \psi + z$ is hydraulic head [L], ψ is gage pressure head [L], and z is elevation above an arbitrary datum [L]. Darcy derived a form equivalent to (1.1) for one-dimensional flow through sand-packed pipes. Dupuit was the first to apply (1.1) to converging flow by combining it with mass conservation $Q = (2\pi rb)\,u$ across a cylindrical shell concentric with the well, leading to

$$Q = K\,(2\pi rb)\,\frac{\partial h}{\partial r}. \tag{1.2}$$

Integrating (1.2) between two radial distances r_1 and r_2 from the pumping well, Dupuit evaluated the confined steady-state head difference between the two points as

$$h(r_2) - h(r_1) = \frac{Q}{2\pi Kb}\log\left(\frac{r_2}{r_1}\right). \tag{1.3}$$

This is the solution for flow to a well at the center of a circular island, where a constant head condition is applied at the edge of the island (r_2).

Dupuit (1857) also derived a radial flow solution for unconfined aquifers by neglecting the vertical component of flow. Following a similar approach as for confined aquifers, Dupuit (1857) estimated the steady-state head difference between two distances from the pumping well for unconfined aquifers as

$$h^2(r_2) - h^2(r_1) = \frac{Q}{\pi K} \log\left(\frac{r_2}{r_1}\right). \tag{1.4}$$

These two solutions are only strictly valid for finite domains; when applied to domains without a physical boundary at r_2, the outer radius essentially becomes a fitting parameter. The solutions are also used in radially infinite systems under pseudo-static conditions, when the shape of the water table does not change with time.

Equations (1.3) and (1.4) are equivalent when b in (1.3) is average head $(h(r_1) + h(r_2))/2$. In developing (1.4), Dupuit (1857) used the following assumptions (now commonly called the Dupuit assumptions) in context of unconfined aquifers:

- the aquifer bottom is a horizontal plane;
- groundwater flow toward the pumping wells is horizontal with no vertical hydraulic gradient component;
- the horizontal component of the hydraulic gradient is constant with depth and equal to that slope of the water table; and
- there is no seepage face at the borehole.

These assumptions are one of the main approaches to simplifying the unconfined flow problem and making it analytically tractable. In the unconfined flow problem both the head and the location of the water table are unknowns; the Dupuit assumptions eliminate one of the unknowns.

1.2.2 Historical Developments after Dupuit

de Vries (2007) gives a detailed historical account of groundwater hydrology; only history relevant to well test analysis is given here. Forchheimer (1886) first recognized the Laplace equation $\nabla^2 h = 0$ governed two-dimensional steady confined groundwater flow (to which (1.3) is a solution), allowing analogies to be drawn between groundwater flow and the field of steady-state heat conduction, including the first application of conformal mapping to solve a groundwater flow problem. Slichter (1898) also arrived at the Laplace equation for groundwater flow, and was the first to account for a vertical flow component. Utilizing Dupuit's assumptions, Forchheimer (1898) developed the steady-state unconfined differential equation (to which (1.4) is a solution), $\nabla^2 h^2 = 0$. Boussinesq (1904) first gave the transient version of the confined groundwater flow equation $\alpha_s \nabla^2 h = \partial h / \partial t$ (where $\alpha_s = K/S_s$ is hydraulic diffusivity [L^2/T] and S_s is specific storage [1/L]), based upon analogy with transient heat conduction.

In Prague, Thiem (1906) was possibly the first to use (1.3) for estimating K from pumping tests with multiple observation wells (Simmons, 2008). Equation (1.3) (commonly called the Thiem equation) was tested in the 1930's both in the field (Wenzel (1932) performed a 48-hour pumping test with 80 observation wells in Grand Island, Nebraska) and in the laboratory (Wyckoff et al. (1932) developed a 15-degree unconfined wedge sand tank to simulate converging flow). Both found the steady-state solution lacking in ability to consistently estimate aquifer parameters. Wenzel (1942) developed several complex averaging approaches (the "limiting" and "gradient" formulas) to attempt to consistently estimate K using steady-state confined equations for a finite system from transient unconfined data. Muskat (1932) considered partial-penetration effects in steady-state flow to wells, discussing the nature of errors associated with assumption of uniform flux across the well screen in a partially penetrating well. Muskat's textbook on porous media flow (Muskat, 1937) summarized much of what was known in hydrology and petroleum reservoir engineering around the time of the next major advance in well test solutions by Theis.

1.2.3 Confined Transient Flow

Theis (1935) utilized the analogy between transient groundwater flow and heat conduction to develop an analytical solution for confined transient flow to a pumping well (see Figure 1.1). The analytical solution was based on a Green's function heat conduction solution in an infinite axis-symmetric slab due to an instantaneous line heat source or sink (Carslaw, 1921). With the aid of mathematician Clarence Lubin, Theis extended the heat conduction solution to a continuous source, motivated to better explain the results of pumping tests like the 1931 test in Grand Island. Theis (1935) gave an expression for drawdown due to pumping a well at rate Q in a homogeneous, isotropic confined aquifer of infinite radial extent as an exponential integral

$$s(r,t) = \frac{Q}{4\pi T} \int_{r^2/(4\alpha_s t)}^{\infty} \frac{e^{-u}}{u} \, \mathrm{d}u, \tag{1.5}$$

where $s = h_0(r) - h(t,r)$ is drawdown, h_0 is pre-test hydraulic head, $T = Kb$ is transmissivity, and $S = S_s b$ is storativity. Equation (1.5) is a solution to the diffusion equation, with a zero-drawdown inital and far-field condition,

$$s(r, t=0) = s(r \to \infty, t) = 0. \tag{1.6}$$

The pumping well was approximated by a line sink (zero radius), and the source term assigned there was based upon (1.2),

$$\lim_{r \to 0} r \frac{\partial s}{\partial r} = -\frac{Q}{2\pi T}. \tag{1.7}$$

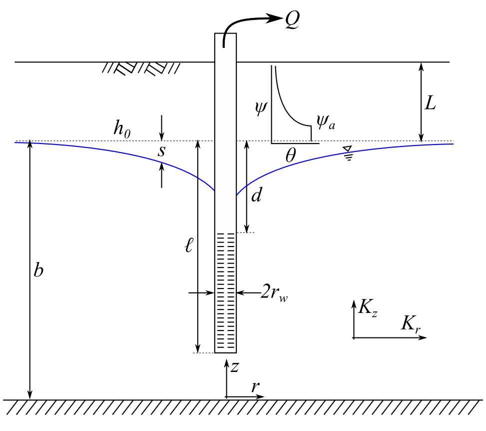

Figure 1.1 Unconfined well test diagram

Although the transient governing equation was known through analogy with heat conduction, the transient storage mechanism (analogous to specific heat capacity) was not completely understood. Unconfined aquifer tests were known to experience slower drawdown than confined tests, due to water supplied by dewatering the zone near the water table, which is related to the formation specific yield (porosity less residual water). Muskat (1934) and Hurst (1934) derived solutions to confined transient radial flow problems for finite domains, but attributed transient storage solely to fluid compressibility. Jacob (1940) derived the diffusion equation for groundwater flow in compressible elastic confined aquifers, using mass conservation and

Darcy's law (without recourse to analogy with heat conduction). Terzaghi (1923) developed the one-dimensional consolidation theory in the related field of soil mechanics, uknown to most hydrologists (Batu, 1998). Meinzer (1928) proposed a confined storage mechanism related to aquifer compaction, and Jacob (1940) formally showed $S_s = \rho_w g(\beta_p + n\beta_w)$, where ρ_w and β_w are fluid density [M/L^3] and compressibility [M/(T^{2}L)], n is dimensionless porosity, and β_p is formation bulk compressibility. The axis-symmetric diffusion equation in radial coordinates is

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{1}{\alpha_s}\frac{\partial s}{\partial t}. \tag{1.8}$$

When deriving analytical expressions, the governing equation is commonly made dimensionless to simplify presentation of results. For flow to a pumping well, it is convenient to use $L_C = b$ as a characteristic length, $T_C = Sb^2/T$ as a characteristic time, and $H_C = Q/(4\pi T)$ as a characteristic head. The dimensionless diffusion equation is

$$\frac{\partial^2 s_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial s_D}{\partial r_D} = \frac{\partial s_D}{\partial t_D}, \tag{1.9}$$

where $r_D = r/L_C$, $s_D = s/H_c$, and $t_D = t/T_C$ are scaled by characteristic quantities.

The Theis (1935) solution was developed for field application to estimate aquifer hydraulic properties, but it saw limited use because it was difficult to compute the exponential integral for arbitrary inputs. Wenzel (1942) proposed a type-curve method that enabled graphical application of the Theis (1935) solution to field data. Cooper and Jacob (1946) suggested that for large values of t_D ($t_D \geq 25$), the infinite integral in the Theis (1935) solution can be approximated as

$$s_D(t_D, r_D) = \int_{r^2/(4\alpha_s t)}^{\infty} \frac{e^{-u}}{u}\,\mathrm{d}u \approx \log_e\left(\frac{4Tt}{r^2 S}\right) - \gamma \tag{1.10}$$

where $\gamma \approx 0.57722$ is the Euler-Mascheroni constant. This leads to Jacob and Cooper's straight-line simplification

$$\Delta s \approx 2.3\frac{Q}{4\pi T} \tag{1.11}$$

where Δs is the drawdown over one log-cycle (base 10) of time. The intercept of the straight-line approximation is related to S through (1.10) This approximation made estimating hydraulic parameters much simpler at large t_D. Hantush (1961) later extended Theis' confined solution for partially penetrating wells.

1.2.4 Observed Time-drawdown Curve

Before the time-dependent solution of Theis (1935), distance drawdown was the diagnostic plot for aquifer test data. Detailed distance-drawdown plots require many observation locations (e.g., the 80 observation wells of Wenzel (1936)). Re-analyzing results of the unconfined pumping test in Grand Island, Wenzel (1942) noticed that the Theis (1935) solution gave inconsistent estimates of S and K, attributed to the delay in the yield of water from storage as the water table fell. The Theis (1935) solution corresponds to the Dupuit assumptions for unconfined flow, and can only re-create the a portion of observed unconfined time-drawdown profiles (either late or early). The effect of the water table must be taken into account through a boundary condition or source term in the governing equation to reproduce observed behavior in unconfined pumping tests.

Walton (1960) recognized three distinct segments characterizing different release mechanisms on time-drawdown curve under water table conditions (see Figure 1.2). A log-log time-drawdown

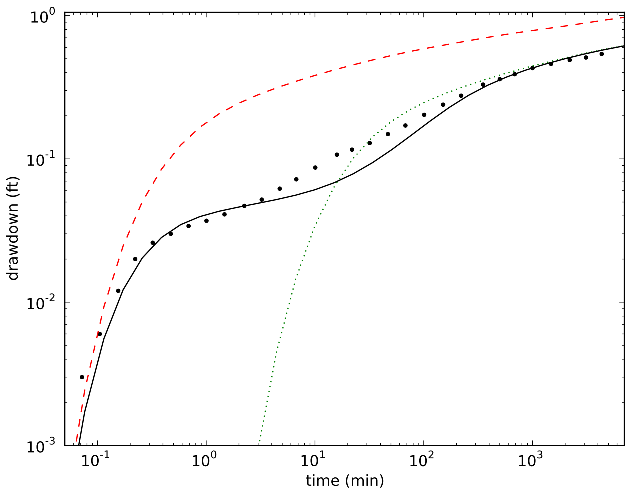

Figure 1.2 Drawdown data from Cape Cod (Moench et al., 2001), observation well F377-037. Upper dashed curve is confined model of Hantush (1961) with $S = S_s b$, lower dotted curve is same with $S = S_s b + S_y$. Solid curve is unconfined model of Neuman (1974) using $S_y = 0.23$.

plot in an unconfined aquifer has a characteristic shape consisting of a steep early-time segment, a flatter intermediate segment and a steeper late-time segment. The early segment behaves like the Theis (1935) solution with $S = S_s b$ (water release due to bulk medium relaxation), the late segment behaves like the Theis (1935) solution with $S = S_s b + S_y$ (Gambolati, 1976) (water release due to water table drop), and the intermediate segment represents a transition between the two. Distance-drawdown plots from unconfined aquifer tests do not show a similar inflection or change in slope, and do not produce good estimates of storage parameters.

1.3 Early Unconfined Well Test Solutions

1.3.1 Moving Water Table Solutions Without Confined Storage

The Theis (1935) solution for confined aquifers can only reproduce either the early or late segments of the unconfined time-drawdown curve (see Figure 1.2). Boulton (1954a) suggested it is theoretically unsound to use the Theis (1935) solution for unconfined flow because it does not account for vertical flow to the pumping well. He proposed a new mechanism for flow towards a fully penetrating pumping well under unconfined conditions. His formulation assumed flow is governed by $\nabla^2 s = 0$, with transient effects incorporated through the water table boundary condition. He treated the water table (where $\psi = 0$, located at $z = \xi$ above the base of the aquifer) as a moving material boundary subject to the condition $h\,(r, z = \xi, t) = z$. He considered the water table without recharge to be comprised of a constant set of particles, leading to the kinematic boundary condition

$$\frac{D}{Dt}\,(h - z) = 0 \tag{1.12}$$

which is a statement of conservation of mass, for an incompressible fluid. Boulton (1954a) considered the Darcy velocity of the water table as $u_z = -\frac{K_z}{S_y}\frac{\partial h}{\partial z}$ and $u_r = -\frac{K_r}{S_y}\frac{\partial h}{\partial r}$, and expressed the total derivative as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} - \frac{K_r}{S_y}\frac{\partial h}{\partial r}\frac{\partial}{\partial r} - \frac{K_z}{S_y}\frac{\partial h}{\partial z}\frac{\partial}{\partial z}, \tag{1.13}$$

where K_r and K_z are radial and vertical components hydraulic conductivity. Using (1.13), the kinematic boundary condition (1.12) in terms of drawdown is

$$\frac{\partial s}{\partial t} - \frac{K_r}{S_y}\left(\frac{\partial s}{\partial r}\right)^2 - \frac{K_z}{S_y}\left(\frac{\partial s}{\partial z}\right)^2 = -\frac{K_z}{S_y}\frac{\partial s}{\partial z}. \tag{1.14}$$

Boulton (1954a) utilized the wellbore and far-field boundary conditions of Theis (1935). He also considered that the aquifer rests on an impermeable flat horizontal boundary $\partial h/\partial z|_{z=0} = 0$; this was inferred by Theis (1935) because of his two-dimensional radial flow assumption. Dagan (1967) extended Boulton's water table solution to the partially penetrating case by replacing the wellbore boundary condition with

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = \begin{cases} \frac{Q}{2\pi K(\ell - d)} & b-\ell < z < b-d \\ 0 & \text{otherwise} \end{cases}, \tag{1.15}$$

where ℓ and d are the upper and lower boundaries of the pumping well screen, as measured from the initial top of the aquifer.

The two sources of non-linearity in the unconfined problem are: 1) the boundary condition is applied at the water table, the location of which is unknown *a priori*; 2) the boundary condition applied on the water table includes h^2 terms.

In order to solve this non-linear problem both Boulton and Dagan linearized it by disregarding second order components in the free-surface boundary condition (1.14) and forcing the free surface to stay at its initial position, yielding

$$\frac{\partial s}{\partial t} = -\frac{K_z}{S_y}\frac{\partial s}{\partial z} \qquad z = h_0, \tag{1.16}$$

which now has no horizontal flux component after neglecting second-order terms. Equation (1.16) can be written in non-dimensional form as

$$\frac{\partial s_D}{\partial t_D} = -K_D \sigma^* \frac{\partial s_D}{\partial z_D} \qquad z_D = 1, \tag{1.17}$$

where $K_D = K_z/K_r$ is the dimensionless anisotropy ratio and $\sigma^* = S/S_y$ is the dimensionless storage ratio.

Both Boulton (1954a) and Dagan (1967) solutions reproduce the intermediate and late segments of the typical unconfined time-drawdown curve, but neither of them reproduces the early segment of the curve. Both are solutions to the Laplace equation, and therefore disregard confined aquifer storage, causing pressure pulses to propagate instantaneously through the saturated zone. Both solutions exhibit an instantaneous step-like increase in drawdown as soon as the pumping starts.

1.3.2 Delayed Yield Unconfined Response

Boulton (1954b) extended Theis' transient confined theory to include the effect of delayed yield due to movement of the water table in unconfined aquifers. Boulton's proposed solutions (1954b; 1963) reproduce all three segments of the unconfined time-drawdown curve. In his formulation of delayed yield, he assumed that as the water table falls water is released from storage (through drainage) gradually, rather than instantaneously as in the free-surface solutions of Boulton (1954a) and Dagan (1967). This approach yielded an integro-differential flow equation in terms of vertically averaged drawdown s^* as

$$\frac{\partial^2 s^*}{\partial r^2} + \frac{1}{r}\frac{\partial s^*}{\partial r} = \left[\frac{S}{T}\frac{\partial s^*}{\partial t}\right] + \left\{\alpha S_y \int_0^t \frac{\partial s^*}{\partial \tau} e^{-\alpha(t-\tau)}\,\mathrm{d}\tau\right\} \tag{1.18}$$

which Boulton linearized by treating T as constant. The term in square brackets is instantaneous confined storage, the term in braces is a convolution integral representing storage released gradually since pumping began, due to water table decline. Boulton (1963) showed the time when delayed yield effects become negligible is approximately equal to $\frac{1}{\alpha}$, leading to the term "delay index". Prickett (1965) used this concept and through analysis of large amount of field drawdown data with Boulton (1963) solution, he established an empirical relationship between the delay index and physical aquifer properties. Prickett proposed a methodology for estimation of S, S_y, K, and α of unconfined aquifers by analyzing pumping tests with the Boulton (1963) solution.

Although Boulton's model was able to reproduce all three segment of the unconfined time-drawdown curve, it failed to explain the physical mechanism of the delayed yield process because of the non-physical nature of the "delay index" in the Boulton (1963) solution.

Streltsova (1972a) developed an approximate solution for the decline of the water table and s^* in fully penetrating pumping and observation wells. Like Boulton (1954b), she treated the water table as a sharp material boundary, writing the two-dimensional depth-averaged flow equation as

$$\frac{\partial^2 s^*}{\partial r^2} + \frac{1}{r}\frac{\partial s^*}{\partial r} = \frac{S}{T}\left(\frac{\partial s^*}{\partial t} - \frac{\partial \xi}{\partial t}\right). \tag{1.19}$$

The rate of water table decline was assumed to be linearly proportional to the difference between the water table elevation ξ and the vertically averaged head $b - s^*$,

$$\frac{\partial \xi}{\partial t} = \frac{K_z}{S_y b_z}\left(s^* - b + \xi\right) \tag{1.20}$$

where $b_z = b/3$ is an effective aquifer thickness over which water table recharge is distributed into the deep aquifer. Equation (1.20) can be viewed as an approximation to the zero-order linearized free-surface boundary condition (1.16) of Boulton (1954a) and Dagan (1967). Streltsova considered the initial condition $\xi(r, t = 0) = b$ and used the same boundary condition at the pumping well and the outer boundary ($r \to \infty$) used by Theis (1935) and Boulton (1963). Equation (1.19) has the solution (Streltsova, 1972b)

$$\frac{\partial \xi}{\partial t} = -\alpha_T \int_0^t e^{-\alpha_T (t-\tau)} \frac{\partial s^*}{\partial \tau}\, \mathrm{d}\tau \tag{1.21}$$

where $\alpha_T = K_z/(S_y b_z)$. Substituting (1.21) into (1.20) yields solution (1.18) of Boulton (1954b, 1963); the two solutions are equivalent. Boulton's delayed yield theory (like that of Streltsova) does not account for flow in unsaturated zone but instead treats water table as material boundary moving vertically downward under influence of gravity. Streltsova (1973) used field data collected by Meyer (1962) to demonstrate unsaturated flow had virtually no impact on the observed delayed process. Although Streltsova's solution related Boulton's delay index to physical aquifer properties, it was later found to be a function of r (Neuman, 1975; Herrera et al., 1978). The delayed yield solutions of Boulton and Streltsova do not account for vertical flow within the unconfined aquifer through simplifying assumptions; they cannot be extended to account for partially penetrating pumping and observation wells.

Prickett's pumping test in the vicinity of Lawrenceville, Illinois (Prickett, 1965) showed that specific storage in unconfined aquifers can be much greater than typically observed values in confined aquifers – possibly due to entrapped air bubbles or poorly consolidated shallow sediments. It is clear that the elastic properties of unconfined aquifers are too important to be disregarded.

1.3.3 Delayed Water Table Unconfined Response

Boulton's (1954b; 1963) models encountered conceptual difficulty explaining the physical mechanism of water release from storage in unconfined aquifers. Neuman (1972) presented a physically based mathematical model that treated the unconfined aquifer as compressible (like Boulton (1954b, 1963) and Streltsova (1972a,b)) and the water table as a moving material boundary (like Boulton (1954a) and Dagan (1967)). In Neuman's approach delayed response of aquifer was caused by physical water table movement, he therefore proposed to replace the phrase "delayed yield" by "delayed water table response".

Neuman (1972) replaced the Laplace equation of Boulton (1954a) and Dagan (1967) by the diffusion equation; in dimensionless form it is

$$\frac{\partial^2 s_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial s_D}{\partial r_D} + K_D \frac{\partial^2 s_D}{\partial z_D^2} = \frac{\partial s_D}{\partial t_D}. \tag{1.22}$$

Like Boulton (1954a) and Dagan (1967), Neuman treated the water table as a moving material boundary, linearized it (using (1.17)), and treated the anisotropic aquifer as three-dimensional axis-symmetric. Like Dagan (1967), Neuman (1974) accounted for partial penetration. By including confined storage in the governing equation (1.22), Neuman was able to reproduce all three parts of the observed unconfined time-drawdown curve and produce parameter estimates (including the ability to estimate K_z) very similar to the delayed yield models.

Compared to the delay index models, Neuman's solution produced similar fits to data (often underestimating S_y, though), but Neuman (1975, 1979) questioned the physical nature of Boulton's delay index. He performed a regression fit between the Boulton (1954b) and Neuman (1972) solutions, resulting in the relationship

$$\alpha = \frac{K_z}{S_y b}\left[3.063 - 0.567 \log\left(\frac{K_D r^2}{b^2}\right)\right] \tag{1.23}$$

demonstrating α decreases linearly with $\log r$ and is therefore not a characteristic constant of the aquifer. When ignoring the logarithmic term in (1.23) the relationship $\alpha = 3K_z/(S_y b)$ proposed by Streltsova (1972a) is approximately recovered.

After comparative analysis of various methods for determination of specific yield, Neuman (1987) concluded that water table response to pumping is a much faster phenomenon than drainage in unsaturated zone above it.

Malama (2011) recently proposed an alternative linearization of (1.14), approximately including the effects of the neglected second-order terms, leading to the alternative water table boundary condition of

$$S_y \frac{\partial s}{\partial t} = -K_z\left(\frac{\partial s}{\partial z} + \beta\frac{\partial^2 s}{\partial z^2}\right) \qquad z = h_0 \tag{1.24}$$

where β is a linearization coefficient [L]. The parameter β provides additional adjustment of the shape of the intermediate portion of the time-drawdown curve (beyond adjustments possible with K_D and σ^* alone), leading to improved estimates of S_y. When $\beta = 0$ (1.24) simplifies to (1.16).

1.3.4 Hybrid Water Table Boundary Condition

The solution of Neuman (1972, 1974) was accepted by many hydrologists "as the preferred model ostensibly because it appears to make the fewest simplifying assumptions" (Moench et al., 2001). Despite acceptance, Nwankwor et al. (1984) and Moench (1995) pointed out that significant difference might exist between measured and model-predicted drawdowns, especially at locations near the water table, leading to significantly underestimated S_y using Neuman's models. Moench (1995) attributed the inability of Neuman's models to give reasonable estimates of S_y and capture this observed behavior near the water table due to the later's disregard of "gradual drainage". In an attempt to resolve this problem, Moench (1995) replaced the instantaneous moving water table boundary condition used by Neuman with one containing a Boulton (1954b) delayed yield convolution integral,

$$\int_0^t \frac{\partial s}{\partial \tau} \sum_{m=1}^{M} \alpha_m e^{-\alpha_m (t-\tau)} \, \mathrm{d}\tau = -\frac{K_z}{S_y} \frac{\partial s}{\partial z} \tag{1.25}$$

This hybrid boundary condition ($M = 1$ in Moench (1995)) included the convolution source term Boulton (1954b, 1963) and Streltsova (1972a,b) used in their depth-averaged governing flow equations. In addition to this new boundary condition, Moench (1995) included a finite radius pumping well with wellbore storage, conceptually similar to how Papadopulos and Cooper Jr. (1967) modified the solution of Theis (1935). In all other respects, his definition of the problem was similar to that of Neuman (1974).

Moench's solution resulted in improved fits to experimental data and produced more realistic estimates of specific yield (Moench et al., 2001), including the use of multiple delay parameters α_m (Moench, 2003). Moench et al. (2001) used (1.25) with $M = 3$ to estimate hydraulic parameters in the unconfined aquifer at Cape Cod. They showed that $M = 3$ enabled a better fit to the observed drawdown data than obtained by $M < 3$ or the model of Neuman (1974). Similar to the parameter α in Boulton's model, the physical meaning of α_m are not clear.

1.4 Unconfined Solutions Considering Unsaturated Flow

As an alternative to linearizing the water table condition of Boulton (1954a), the unsaturated zone can be explicitly included. The non-linearity of unsaturated flow is substituted for the non-linearity of (1.14). By considering the vadose zone, the water table is internal to the domain, rather than a boundary condition. The model-data misfit in Figure 1.2 at "late intermediate" time is one of the motivations for considering the mechanisms of delayed yield and the effects of the unsaturated zone.

1.4.1 Unsaturated Flow Without Confined Aquifer Storage

Kroszynski and Dagan (1975) were the first to account analytically for the effect of the unsaturated zone on aquifer drawdown. They extended the solution of Dagan (1967) by accounting for unsaturated flow above the water table. They used Richards' equation for axis-symmetric unsaturated flow in a vadose zone of thickness L

$$K_r \frac{1}{r}\frac{\partial}{\partial r}\left(k(\psi) r \frac{\partial \sigma}{\partial r}\right) + K_z \frac{\partial}{\partial z}\left(k(\psi)\frac{\partial \sigma}{\partial z}\right) = C(\psi)\frac{\partial \sigma}{\partial t} \qquad \xi < z < b + L \tag{1.26}$$

where $\sigma = b + \psi_a - h$ is unsaturated zone drawdown [L], ψ_a is air-entry pressure head [L], $0 \leq k(\psi) \leq 1$ is dimensionless relative hydraulic conductivity, $C(\psi) = d\theta/d\psi$ is moisture retention curve [1/L], and θ is dimensionless volumetric water content (see inset in Figure 1.1). They assumed flow in the underlying saturated zone was governed by the Laplace equation (like Dagan (1967)). The saturated and unsaturated flow equations were coupled through interface conditions at the water table expressing continuity of hydraulic heads and normal groundwater fluxes,

$$s = \sigma \qquad \nabla s \cdot \mathbf{n} = \nabla \sigma \cdot \mathbf{n} \qquad z = \xi \tag{1.27}$$

where $\mathbf{n}$ is the unit vector perpendicular to the water table.

To solve the unsaturated flow equation (1.26), Kroszynski and Dagan (1975) linearized (1.26) by adopting the Gardner (1958) exponential model for the relative hydraulic conductivity, $k(\psi) = e^{\kappa_a(\psi-\psi_a)}$, where κ_a is the sorptive number [1/L] (related to pore size). They adopted the same exponential form for the moisture capacity model, $\theta(\psi) = e^{\kappa_k(\psi-\psi_k)}$, where ψ_k is the pressure at which $k(\psi) = 1$, $\kappa_a = \kappa_k$, and $\psi_a = \psi_k$, leading to the simplified form $C(\psi) = S_y\kappa_a e^{\kappa_a(\psi-\psi_a)}$. In the limit as $\kappa_k = \kappa_a \to \infty$ their solution reduces to that of Dagan (1967). The relationship between pressure head and water content is a step function. Kroszynski and Dagan (1975) took unsaturated flow above the water table into account but ignored the effects of confined aquifer storage, leading to similar early-time step-change behavior to Boulton (1954a) and Dagan (1967).

1.4.2 Increasingly Realistic Saturated-Unsaturated Well Test Models

Mathias and Butler (2006) combined the confined aquifer flow equation (1.22) with a one-dimensional linearized version of (1.26) for a vadose zone of finite thickness. Their water table was treated as a fixed boundary with known flow conditions, decoupling the unsaturated and saturated solutions at the water table. Although they only considered a one-dimensional unsaturated zone, they included the additional flexibility provided by different exponents ($\kappa_a \neq \kappa_k$) and reference pressure heads ($\psi_a \neq \psi_k$). Mathias and Butler (2006) did not consider a partially penetrating well, but they did note the possibility of accounting for it in principle by incorporating their uncoupled drainage function in the solution of Moench (1997), which considers a partially penetrating well of finite radius.

Tartakovsky and Neuman (2007) similarly combined the confined aquifer flow equation (1.22), but with the original axis-symmetric form of (1.26) considered by Kroszynski and Dagan (1975). Also like Kroszynski and Dagan (1975), their unsaturated zone was characterized by a single exponent $\kappa_a = \kappa_k$ and reference pressure head $\psi_a = \psi_k$. Unlike Kroszynski and Dagan (1975) and Mathias and Butler (2006), Tartakovsky and Neuman (2007) assumed an infinitely thick unsaturated zone.

Tartakovsky and Neuman (2007) demonstrated flow in the unsaturated zone is not strictly vertical. Numerical simulations by Moench (2008) showed groundwater movement in the capillary fringe is more horizontal than vertical. Mathias and Butler (2006) and Moench (2008) showed that using the same exponents and reference pressure heads for effective saturation and relative permeability decreases model flexibility and underestimates S_y. Moench (2008) predicted an extended form of Tartakovsky and Neuman (2007) with two separate exponents, a finite unsaturated zone, and wellbore storage would likely produce more physically realistic estimates of S_y.

Mishra and Neuman (2010) developed a new generalization of the solution of Tartakovsky and Neuman (2007) that characterized relative hydraulic conductivity and water content using $\kappa_a \neq \kappa_k$, $\psi_a \neq \psi_k$ and a finitely thick unsaturated zone. Mishra and Neuman (2010) validated their solution against numerical simulations of drawdown in a synthetic aquifer with unsaturated properties given by the model of van Genuchten (1980). They also estimated aquifer parameters from Cape Cod drawdown data (Moench et al., 2001), comparing estimated van Genuchten (1980) parameters with laboratory values (Mace et al., 1998).

Mishra and Neuman (2011) further extended their 2010 solution to include a finite-diameter pumping well with storage. Mishra and Neuman (2010, 2011) were the first to estimate non-exponential model unsaturated aquifer properties from pumping test data, by curve-fitting the exponential model to the van Genuchten (1980) model. Analyzing pumping test data of Moench et al. (2001) (Cape Cod, Massachusetts) and Nwankwor et al. (1984, 1992) (Borden, Canada), they estimated unsaturated flow parameters similar to laboratory-estimated values for the same soils.

1.5 Future Challenges

The conceptualization of groundwater flow during unconfined pumping tests has been a challenging task that has spurred substantial theoretical research in the field hydrogeology for decades. Unconfined flow to a well is non-linear in multiple ways, and the application of analytical solutions has required utilization of advanced mathematical tools. There are still many additional challenges to be addressed related to unconfined aquifer pumping tests, including:

- Hysteretic effects of unsaturated flow. Different exponents and reference pressures are needed during drainage and recharge events, complicating simple superposition needed to handle multiple pumping wells, variable pumping rates, or analysis of recovery data.
- Collecting different data types. Validation of existing models and motivating development of more realistic ones depends on more than just saturated zone head data. Other data types include vadose zone water content (Meyer, 1962), and hydrogeophysical data like microgravity (Damiata and Lee, 2006) or streaming potentials (Malama et al., 2009).
- Moving water table position. All solutions since Boulton (1954a) assume the water table is fixed horizontal $\xi(r,t) = h_0$ during the entire test, even close to the pumping well where large drawdown is often observed. Iterative numerical solutions can accommodate this, but this has not been included in an analytical solution.
- Physically realistic partial penetration. Well test solutions suffer from the complication related to the unknown distribution of flux across the well screen. Commonly, the flux distribution is simply assumed constant, but it is known that flux will be higher near the ends of the screened interval that are not coincident with the aquifer boundaries.
- Dynamic water table boundary condition. A large increase in complexity comes from explicitly including unsaturated flow in unconfined solutions. The kinematic boundary condition expresses mass conservation due to water table decline. Including an analogous dynamic boundary condition based on a force balance (capillarity vs. gravity) may include sufficient effects of unsaturated flow, without the complexity associated with the complete unsaturated zone solution.
- Heterogeneity. In real-world tests heterogeneity is present at multiple scales. Large-scale heterogeneity (e.g., faults or rivers) can sometimes be accounted in analytical solutions using the method of images, or other types of superpostion. A stochastic approach (Neuman et al., 2004) could alternatively be developed to estimate random unconfined aquifer parameter distribution parameters.

Despite advances in considering physically realistic unconfined flow, most real-world unconfined tests (e.g., Wenzel (1942), Nwankwor et al. (1984, 1992), or Moench et al. (2001)) exhibit non-classical behavior that deviates from the early-intermediate-late behavior predicted by the models summarized here. We must continue to strive to include physically relevant processes and representatively linearize non-linear phenomena, to better understand, simulate and predict unconfined flow processes.

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